BRST and Anti-BRST Symmetries in Perturbative Quantum Gravity

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Abstract

In perturbative quantum gravity, the sum of the classical Lagrangian density, a gauge fixing term and a ghost term is invariant under two sets of supersymmetric transformations called the BRST and the anti-BRST transformations. In this paper we will analyse the BRST and the anti-BRST symmetries of perturbative quantum gravity in curved spacetime, in linear as well as non-linear gauges. We will show that even though the sum of ghost term and the gauge fixing term can always be expressed as a total BRST or a total anti-BRST variation, we can express it as a combination of both of them only in certain special gauges. We will also analyse the violation of nilpotency of the BRST and the anti-BRST transformations by introduction of a bare mass term, in the massive Curci-Ferrari gauge.

Key words: BRST, Anti-BRST, Perturbative quantum gravity
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1 Introduction

Three out of the four fundamental forces in nature are described by Yang-Mills theories. The fourth, being gravity, is described by gauge theory of diffeomorphism [1]. In this sense all the forces of nature can be formulated in the language of gauge theory.

However, when analysing with any gauge theory, we have to deal with the redundant degrees of freedom due to gauge invariance of that theory. We have to eliminate these redundant degrees of freedom before trying to quantize that theory. An elegant formalism called the BRST formalism is usually employed for this purpose [2]. In this formalism the sum of the classical Lagrangian density, a gauge fixing term and a ghost term (collectively called a gauge fixing Lagrangian density in this paper) is invariant under a set of supersymmetric transformations called the BRST transformations. This total Lagrangian density is also invariant under another set of supersymmetric transformations called the anti-BRST transformations [3].

The BRST and the anti-BRST symmetries for perturbative quantum gravity in four dimensional flat spacetime have been studied by a number of authors

[4-6] and their work has been summarized by N. Nakanishi and I. Ojima [7]. The BRST symmetry in two dimensional curved spacetime has been thoroughly studied [8-10]. The BRST and the anti-BRST symmetries for topological quantum gravity in curved spacetime have also been studied [11-12]. All this work has been done in linear gauges.

However BRST and anti-BRST symmetries are known to have a richer structure in Yang-Mills theories. In case of Yang-Mills theories, it is known that in Landau gauge we can express the gauge fixing Lagrangian density as a combination of total BRST and total anti-BRST variations [13]. This is also achieved by addition of suitable non-linear terms to the gauge fixing Lagrangian density [14]. Furthermore, the addition of a bare mass term breaks the nilpotency of the BRST and the anti-BRST transformations and this leads to the violation of unitarity of the resultant theory [15].

In this paper we will try to generalize these results that are known in the context of Yang-Mills theories in four dimensional flat spacetime to perturbative quantum gravity in curved spacetime, in arbitrary dimensions. It may be noted that the violation of unitarity did not have much physical relevance in the context of Yang-Mills theories. However it is suspected that certain quantum gravitational processes might lead to violation of unitarity [16]. So this loss of unitarity, due to the addition of a bare mass term, seems to be physically more relevant to quantum gravity in curved spacetime than Yang-Mills theories in flat spacetime.

2 BRST and Anti-BRST Transformations

The Lagrangian density for pure Euclidean gravity with cosmological constant λ is given by

$$\mathcal{L} = \sqrt{g}(R - 2\lambda), \tag{1}$$

where we have adopted units, such that

$$16\pi G = 1. \tag{2}$$

In perturbative gravity one splits the full metric g^f_{ab} into the metric for the background spacetime g_{ab} and a small perturbation around it being h_{ab}. The covariant derivatives along with the lowering and raising of indices are compatible with the metric for the background spacetime. The small perturbation h_{ab} is viewed as the field that is to be quantized.

All the degrees of freedom in h_{ab} are not physical as the Lagrangian density for it is invariant under a gauge transformation,

$$\delta_\Lambda h_{ab} = \nabla_a \Lambda_b + \nabla_b \Lambda_a + \pounds_{(\Lambda)} h_{ab}, \tag{3}$$

where

$$\pounds_{(\Lambda)} h_{ab} = \Lambda^c \nabla_c h_{ab} + h_{ac} \nabla_b \Lambda^c + h_{cb} \nabla_a \Lambda^c. \tag{4}$$

is the Lie derivative of h_{ab} with respect to the vector field Λ^a.

These unphysical degrees of freedom give rise to constraints [17] in the canonical quantization and divergences in the partition function [18] in the path integral quantization. So before we can quantize this theory, we need to fix a gauge

by adding a gauge fixing term. In order to ensure unitarity, a ghost term also has to be added.

Now we first start with the following gauge fixing condition,

$$G[h]_a = (\nabla^b h_{ab} - k\nabla_a h) = 0, \tag{5}$$

where

$$k \neq 1. \tag{6}$$

For $k = 1$, the constraints are not removed and the partition function again diverges. That is why, k is usually written as $1 + \beta^{-1}$, where β is an arbitrary finite constant [19].

The gauge fixing term corresponding to this gauge fixing condition is given by

$$\mathcal{L}_{gf} = \sqrt{g}\left[ib^a(\nabla^b h_{ab} - k\nabla_a h) + \frac{\alpha}{2}b^a b_a\right], \tag{7}$$

and the ghost term is given by

$$\mathcal{L}_{gh} = i\sqrt{g}\,\overline{c}^a\nabla^b[\nabla_a c_b + \nabla_b c_a - 2kg_{ab}\nabla_c c^c + \pounds_{(c)}h_{ab} - kg_{ab}g^{cd}\pounds_{(c)}h_{cd}], \tag{8}$$

where $\pounds_{(c)}h_{ab}$ is the Lie derivative of h_{ab} with respect to the ghost field c^a,

$$\pounds_{(c)}h_{ab} = c^c\nabla_c h_{ab} + h_{ac}\nabla_b c^c + h_{cb}\nabla_a c^c. \tag{9}$$

This ghost term can be expressed as

$$\mathcal{L}_{gh} = \sqrt{g}\,\overline{c}^a M_{ab} c^b, \tag{10}$$

where M_{ab} is given by

$$\begin{aligned} M_{ab} &= i\nabla_c[\delta^c_b\nabla_a + g_{ab}\nabla^c - 2k\delta^c_a\nabla_b + (\nabla_b h^c_a) - h_{ab}\nabla^c + h^c_b\nabla_a \\ &\quad -kg^c_a g^{ef}((\nabla_b h_{ef}) + h_{eb}\nabla_f + h_{fb}\nabla_e)]. \end{aligned} \tag{11}$$

The total Lagrangian density obtained by addition of the original classical Lagrangian, the gauge fixing term and the ghost term is invariant under the following BRST transformations,

$$\begin{aligned} s\,h_{ab} &= \nabla_a c_b + \nabla_b c_a + \pounds_{(c)}h_{ab}, \\ s\,c^a &= -c_b\nabla^b c^a, \\ s\,\overline{c}^a &= b^a, \\ s\,b^a &= 0. \end{aligned} \tag{12}$$

This total Lagrangian density is also invariant under the following anti-BRST transformations,

$$\begin{aligned} \overline{s}\,h_{ab} &= \nabla_a \overline{c}_b + \nabla_b \overline{c}_a + \pounds_{(\overline{c})}h_{ab}, \\ \overline{s}\,c^a &= -b^a - 2\overline{c}_b\nabla^b c^a, \\ \overline{s}\,\overline{c}^a &= -\overline{c}_b\nabla^b\overline{c}^a, \\ \overline{s}\,b^a &= -b^b\nabla_b c^a. \end{aligned} \tag{13}$$

These BRST and anti-BRST transformations appear very different. However, for Yang-Mills theories, the BRST and the anti-BRST transformations

almost seem to reverse their respective roles, by suitably changing the Nakanishi-Lautrup field [7].

We will now analyse this reversing of the form of BRST and anti-BRST transformations for perturbative quantum gravity. To do so, we first shift the original Nakanishi-Lautrup field by $2\overline{c}^b\nabla_b c^a$, and then multiply it by -1, to get a new Nakanishi-Lautrup field.

Then in terms of this new Nakanishi-Lautrup field the BRST transformation are given by

$$\begin{aligned} s\,h_{ab} &= \nabla_a c_b + \nabla_b c_a + \pounds_{(c)} h_{ab}, \\ s\,\overline{c}^a &= -b^a - 2\overline{c}_b \nabla^b c^a, \\ s\,c^a &= -c_b \nabla^b c^a, \\ s\,b^a &= -b^b \nabla_b \overline{c}^a, \end{aligned} \tag{14}$$

and the anti-BRST transformations are given by

$$\begin{aligned} \overline{s}\,h_{ab} &= \nabla_a \overline{c}_b + \nabla_b \overline{c}_a + \pounds_{(\overline{c})} h_{ab}, \\ \overline{s}\,c^a &= b^a, \\ \overline{s}\,\overline{c}^a &= -\overline{c}_b \nabla^b \overline{c}^a, \\ \overline{s}\,b^a &= 0. \end{aligned} \tag{15}$$

These BRST and anti-BRST transformations look like the reversed version of the original BRST and anti-BRST transformations.

Both these sets of transformations are nilpotent. In fact they satisfy,

$$s^2 = \overline{s}^2 = s\overline{s} + \overline{s}s = 0. \tag{16}$$

We can now express the gauge fixing Lagrangian density $\mathcal{L}_g$, which is given by the sum of the gauge fixing term and the ghost term, as follows:

$$\begin{aligned} \mathcal{L}_g &= \mathcal{L}_{gf} + \mathcal{L}_{gh} \\ &= is\sqrt{g}\left[\overline{c}^a(\nabla^b h_{ab} - k\nabla_a h - \frac{i\alpha}{2} b_a)\right] \\ &= -i\overline{s}\sqrt{g}\left[c^a(\nabla^b h_{ab} - k\nabla_a h - \frac{i\alpha}{2} b_a)\right]. \end{aligned} \tag{17}$$

In a slightly different gauge it can also be written, as follows:

$$\begin{aligned} \mathcal{L}_g &= -\frac{i}{2} s\overline{s}\sqrt{g}(h^{ab}h_{ab}) + \frac{i\alpha}{2}\overline{s}\sqrt{g}(b^a c_a) \\ &= \frac{i}{2}\overline{s}s\sqrt{g}(h^{ab}h_{ab}) - \frac{i\alpha}{2} s\sqrt{g}(b^a \overline{c}_a). \end{aligned} \tag{18}$$

Thus the gauge fixing Lagrangian density can be expressed as a total BRST or a total anti-BRST variation.

3 Landau Gauge

In Yang-Mills theories, there is a special gauge called the Landau gauge, in which we can express the gauge fixing Lagrangian density as a combination

of total BRST and total anti-BRST variations. Furthermore, in this gauge the BRST and the anti-BRST variations look similar to each other, with ghosts and anti-ghosts interchanged, and the sign of Nakanishi-Lautrup field also changed [13].

We will now analyse the BRST and anti-BRST symmetry for perturbative quantum gravity in Landau gauge. In Landau gauge, $\alpha = 0$, and so we have

$$\begin{aligned} \mathcal{L}_g &= is\sqrt{g}\left[\overline{c}^a(\nabla^b h_{ab} - k\nabla_a h)\right] \\ &= -i\overline{s}\sqrt{g}\left[c^a(\nabla^b h_{ab} - k\nabla_a h)\right]. \end{aligned} \tag{19}$$

In a slightly different gauge it can also be written, as follows:

$$\begin{aligned} \mathcal{L}_g &= -\frac{i}{2}s\overline{s}\sqrt{g}(h^{ab}h_{ab}) \\ &= \frac{i}{2}\overline{s}s\sqrt{g}(h^{ab}h_{ab}). \end{aligned} \tag{20}$$

Thus in Landau gauge the gauge fixing Lagrangian density for perturbative quantum gravity is also expressed as a combination of a total BRST and a total anti-BRST variation.

In Landau gauge the BRST transformations are given by

$$\begin{aligned} s\,h_{ab} &= \nabla_a c_b + \nabla_b c_a + \pounds_{(c)}h_{ab}, \\ s\,c^a &= -c_b\nabla^b c^a, \\ s\,\overline{c}^a &= b^a, \\ s\,b^a &= 0, \end{aligned} \tag{21}$$

and the anti-BRST transformations are given by

$$\begin{aligned} \overline{s}\,h_{ab} &= \nabla_a \overline{c}_b + \nabla_b \overline{c}_a + \pounds_{(\overline{c})}h_{ab}, \\ \overline{s}\,c^a &= -b^a, \\ \overline{s}\,\overline{c}^a &= -\overline{c}_b\nabla^b \overline{c}^a, \\ \overline{s}\,b^a &= 0. \end{aligned} \tag{22}$$

These transformations look similar to each other, with ghosts and anti-ghosts interchanged, and the sign of Nakanishi-Lautrup field also changed.

4 Non-Linear Gauges

Curci-Ferrari Lagrangian density is non-linear in ghosts and anti-ghosts and thus cannot be obtained directly from the Faddeev-Popov procedure [18]. For Yang-Mills theories in Curci-Ferrari gauge, we can write the gauge fixing Lagrangian density as a combination of total BRST and total anti-BRST variations, for any value of α [14]. In this section we will express the Lagrangian density for perturbative quantum gravity in Curci-Ferrari gauge as a combination of a total BRST and a total anti-BRST variation, for any value of α.

The BRST transformations for perturbative quantum gravity in Curci-Ferrari gauge are given by

$$s\,h_{ab} = \nabla_a c_b + \nabla_b c_a + \pounds_{(c)}h_{ab},$$

$$
\begin{aligned}
s\, c^a &= -c_b \nabla^b c^a, \\
s\, \overline{c}^a &= b^a - \overline{c}^b \nabla_b c^a, \\
s\, b^a &= -b^b \nabla_b c^a - \overline{c}^b \nabla_b c^d \nabla_d c^a,
\end{aligned} \tag{23}
$$

and the anti-BRST transformation for perturbative quantum gravity are given by

$$
\begin{aligned}
\overline{s}\, h_{ab} &= \nabla_a \overline{c}_b + \nabla_b \overline{c}_a + \pounds_{(\overline{c})} h_{ab}, \\
\overline{s}\, \overline{c}^a &= -\overline{c}_b \nabla^b \overline{c}^a, \\
\overline{s}\, c^a &= -b^a - \overline{c}^b \nabla_b c^a, \\
\overline{s}\, b^a &= -b^b \nabla_b \overline{c}^a + c^b \nabla_b \overline{c}^d \nabla_d \overline{c}^a.
\end{aligned} \tag{24}
$$

We can now write a gauge fixing Lagrangian density as a combination of a total BRST and a total anti-BRST variation, as

$$
\begin{aligned}
\mathcal{L}'_g &= \frac{i}{2} s\overline{s} \sqrt{g} \left[h^{ab} h_{ab} - i\alpha \overline{c}^a c_a \right] \\
&= \frac{-i}{2} \overline{s} s \sqrt{g} \left[h^{ab} h_{ab} - i\alpha \overline{c}^a c_a \right].
\end{aligned} \tag{25}
$$

Thus in the Curci-Ferrari gauge, apart from getting the original Faddeev-Popov part of the gauge fixing Lagrangian, we get additional non-linear contributions proportional to $\sqrt{g}(\overline{c}^b \nabla_b c^a)(\overline{c}^d \nabla_d c_a)$. Such terms occur in almost all supersymmetric Yang-Mills theories and string theory. Furthermore, such non-linear terms lead to the formation of off-diagonal ghost-condensates [20]. However, these ghost-condensates do not give rise to any mass term for the gauge fields. This is because the addition of a bare mass term is prevented by the nilpotency of the BRST and the anti-BRST transformations. However, if we are ready to violate the nilpotency of the BRST and the anti-BRST transformations, then we can add such a bare mass term. This has been done for Yang-Mills theories, to get a massive Curci-Ferrari Lagrangian density [15]. Here we will do it for perturbative quantum gravity.

We can write the massive Curci-Ferrari type of Lagrangian density for perturbative quantum gravity as follows:

$$
\begin{aligned}
\mathcal{L}_g^{m^2} &= \frac{i}{2} [s\overline{s} - im^2] \sqrt{g} \left[h^{ab} h_{ab} - i\alpha \overline{c}^a c_a \right] \\
&= \frac{i}{2} [-\overline{s} s - im^2] \sqrt{g} \left[h^{ab} h_{ab} - i\alpha \overline{c}^a c_a \right].
\end{aligned} \tag{26}
$$

Thus the massive Curci-Ferrari type of Lagrangian density for perturbative quantum gravity contains contributions proportional to terms like $\sqrt{g} m^2 g^{ab} g_{ab}$ and $\sqrt{g} i m^2 \alpha \overline{c}^a c_a$.

This Lagrangian density is invariant under the following BRST transformation,

$$
\begin{aligned}
s\, h_{ab} &= \nabla_a c_b + \nabla_b c_a + \pounds_{(c)} h_{ab}, \\
s\, c^a &= -c_b \nabla^b c^a, \\
s\, \overline{c}^a &= b^a - \overline{c}^b \nabla_b c^a, \\
s\, b^a &= im^2 c^a - b^b \nabla_b c^a - \overline{c}^b \nabla_b c^d \nabla_d c^a,
\end{aligned} \tag{27}
$$

and the following anti-BRST transformations,

$$
\begin{aligned}
\overline{s}\, h_{ab} &= \nabla_a \overline{c}_b + \nabla_b \overline{c}_a + \pounds_{(\overline{c})} h_{ab}, \\
\overline{s}\, \overline{c}^a &= -\overline{c}_b \nabla^b \overline{c}^a, \\
\overline{s}\, c^a &= -b^a - \overline{c}^b \nabla_b c^a, \\
\overline{s}\, b^a &= im^2 \overline{c}^a - b^b \nabla_b \overline{c}^a + c^b \nabla_b \overline{c}^d \nabla_d \overline{c}^a.
\end{aligned} \tag{28}
$$

The addition of bare mass term breaks the nilpotency of the BRST and the anti-BRST transformations. The BRST and the anti-BRST transformations now satisfy

$$s^2 = \overline{s}^2 \sim im^2. \tag{29}$$

However, in the zero mass limit, the nilpotency of the BRST and the anti-BRST transformations is restored. This breakdown of nilpotency of the BRST and the anti-BRST transformations also leads to breakdown of the unitarity of the theory. Thus unitarity of this theory is only maintained in the zero mass limit. It is possible that in quantum gravity there could be a breakdown of the unitarity [16] and thus it would be interesting to analyse a formalism that deals with it.

5 Conclusion

In this paper we have generalized certain results from the Yang-Mills theories in flat spacetime to perturbative quantum gravity in curved spacetime. We have shown that the behaviour of BRST and anti-BRST symmetries for perturbative quantum gravity in arbitrary dimensions, in curved spacetime is similar to their behaviour for Yang-Mills theories in four dimensional flat spacetime. Similar to the Yang-Mills theories in flat spacetime, the BRST and the anti-BRST transformations for perturbative quantum gravity almost change their respective forms by a redefinition of the Nakanishi-Lautrup field, in simple linear gauge with an arbitrary value of α. We expressed the gauge fixing Lagrangian density for perturbative quantum gravity as a combination of total BRST and total anti-BRST variations, in Landau gauge. We also expressed the gauge fixing Lagrangian density as a combination of total BRST and total anti-BRST variations, for an arbitrary value of α, by the adding suitable non-linear terms to it. Furthermore, the addition of a bare mass term violated the nilpotency of the BRST and the anti-BRST transformations, which in turn violates the unitarity of the theory. This violation of unitarity could be physically relevant in quantum gravity as it is suspected that certain quantum gravitational processes might lead to a breakdown of the unitarity. We stress the fact that all these results were already known to hold for Yang-Mills theories in flat spacetime and all we have shown here is that they also hold for perturbative quantum gravity in curved spacetime.

6 References

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